

Comment on “Localization of the Maximal Entropy Random Walk”

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Abstract: Burda et al. [Phys. Rev. Lett. 102, 160602 (2009)] asserted that the discrete-time probability distribution $\pi_i(t)$ of the Maximal Entropy Random Walk (MERW) relaxes to the unique stationary profile $\pi_i^* = \psi_{0,i}^2$ as $t \rightarrow \infty$. While π^* is the correct stationary measure and principal-eigenvector localization remains valid, this Comment shows that for any finite connected bipartite graph, the MERW transition matrix possesses a peripheral eigenvalue $\mu = -1$, inducing period-two dynamics. Consequently, the unmodified discrete-time distribution does not converge to π^* from initial conditions with nonzero parity imbalance; instead it approaches a persistent two-cycle. For the unmodified chain, convergence to π^* holds if and only if the initial distribution has equal mass on the two bipartition classes. Cesaro or pairwise parity averaging, aperiodic lazy modifications, and continuous-time formulations recover convergence to π^* in the appropriate sense.

I. Introduction

The Maximal Entropy Random Walk (MERW) constructs stochastic trajectories by maximizing entropy globally over paths of fixed length. For a finite, connected graph $G = (V, E)$ with symmetric adjacency matrix A , spectral radius λ_0 , and normalized principal right eigenvector ψ_0 ($\|\psi_0\|_2 = 1$, $\psi_{0,i} > 0$), the discrete-time stationary distribution is [1]:

$$\pi_i^* = \psi_{0,i}^2$$

Specifically, Ref. [1], following Eq. (2), asserts: “using spectral properties of the matrix P_{ij} , one can show that $\pi_i(t)$ reaches for $t \rightarrow \infty$ a unique stationary state π_i^* obeying (2)”.

For a finite irreducible discrete-time Markov chain, convergence to stationarity from every initial state requires aperiodicity. If P possesses peripheral eigenvalues on the unit circle other than unity, the chain fails to mix. On bipartite topologies commonly studied in MERW simulations (such as ladder networks and square lattices with open or even-periodic boundary conditions), P possesses a peripheral eigenvalue $\mu = -1$. Consequently, while π^* remains stationary, the unmodified discrete-time chain fails to converge from generic initial conditions, settling into a persistent period-two oscillation.

II. Spectral Projection and Row-Stochasticity

Let $G = (V, E)$ denote a finite, connected, bipartite graph with disjoint vertex partitions $V = V_1 \cup V_2$. Ordering vertices conformally yields the symmetric block adjacency matrix:

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

where B is of dimension $|V_1| \times |V_2|$.

Let λ_0 denote the spectral radius of A . By the Perron-Frobenius theorem, λ_0 is a simple, positive eigenvalue, and the corresponding principal right eigenvector ψ_0 satisfies $\psi_{0,i} > 0$ for all $i \in V$. Conformal partitioning of $\psi_0 = (\psi_0^{(1)}, \psi_0^{(2)})^T$ under $A\psi_0 = \lambda_0\psi_0$ yields the coupled linear system:

$$B\psi_0^{(2)} = \lambda_0\psi_0^{(1)}, \quad B^T\psi_0^{(1)} = \lambda_0\psi_0^{(2)}$$

Defining the spectral test vector $\psi_- = (\psi_0^{(1)}, -\psi_0^{(2)})^T$, direct matrix multiplication gives:

$$A\psi_- = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \psi_0^{(1)} \\ -\psi_0^{(2)} \end{pmatrix} = \begin{pmatrix} -B\psi_0^{(2)} \\ B^T\psi_0^{(1)} \end{pmatrix} = \begin{pmatrix} -\lambda_0\psi_0^{(1)} \\ \lambda_0\psi_0^{(2)} \end{pmatrix} = -\lambda_0\psi_-$$

Thus, $-\lambda_0$ is an eigenvalue of A with eigenvector ψ_- . Because the sign-flip mapping on V_2 components provides an invertible isometry between eigenspaces, simplicity of λ_0 implies simplicity of $-\lambda_0$.

The discrete-time MERW transition matrix is defined via diagonal similarity transformation:

$$P = \frac{1}{\lambda_0} D^{-1} A D$$

where $D = \text{diag}(\psi_0)$. Under row probability conventions, P is strictly row-stochastic:

$$\sum_j P_{ij} = \frac{1}{\lambda_0\psi_{0,i}} \sum_j A_{ij}\psi_{0,j} = \frac{(A\psi_0)_i}{\lambda_0\psi_{0,i}} = 1$$

Because A is real and symmetric, A is orthogonally diagonalizable, Jordan blocks do not occur, and spectra satisfy $\text{Spec}(P) = \frac{1}{\lambda_0} \text{Spec}(A)$. Since the peripheral spectrum of A on a connected bipartite graph is strictly $\{\lambda_0, -\lambda_0\}$, the peripheral spectrum of P is strictly $\{1, -1\}$.

The right eigenvector v_2 of P corresponding to the peripheral eigenvalue $\mu_2 = -1$ ($Pv_2 = -v_2$) is the parity vector:

$$v_2 = D^{-1}\psi_- \implies v_{2,i} = \frac{\psi_{-,i}}{\psi_{0,i}} = \begin{cases} +1 & \text{if } i \in V_1 \\ -1 & \text{if } i \in V_2 \end{cases}$$

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This yields an immediate parity diagnostic observable:

$$\pi(t)v_2 = \pi(0)P^t v_2 = (-1)^t \pi(0)v_2 = (-1)^t c$$

where $c = \sum_{i \in V_1} \pi_i(0) - \sum_{i \in V_2} \pi_i(0)$ is the initial parity imbalance. Since $\pi^* v_2 = \sum_{i \in V_1} \psi_{0,i}^2 - \sum_{i \in V_2} \psi_{0,i}^2 = \psi_0^T \psi_- = 0$, convergence $\pi(t) \rightarrow \pi^*$ requires $(-1)^t c \rightarrow 0$, which is impossible whenever $c \neq 0$.

III. Dual Basis Expansion and Asymptotic Oscillations

Let $\pi(t)$ represent the state probability row vector under iterated applications of P . We decompose an arbitrary initial probability distribution $\pi(0)$ into the dual basis of left and right eigenvectors $\{u_\alpha^T, v_\alpha\}$ of P , satisfying the biorthogonality condition $u_\alpha^T v_\beta = \delta_{\alpha\beta}$.

A. Subspace Biorthogonality Verification

Let the stationary mode ($\mu_1 = 1$) be defined by $u_1^T = \pi^*$ and $v_1 = 1$, where $\pi_i^* = \psi_{0,i}^2$ subject to $\sum_i \psi_{0,i}^2 = 1$. Let the oscillatory mode ($\mu_2 = -1$) have right eigenvector v_2 and left eigenvector $u_2^T = w^T$, where $w_i = \psi_{0,i} \psi_{-,i}$ (satisfying $w^T P = -w^T$ via MERW reversibility with respect to π^*). Element-wise:

$$w_i = \begin{cases} \psi_{0,i}^2 & \text{if } i \in V_1 \\ -\psi_{0,i}^2 & \text{if } i \in V_2 \end{cases}$$

Eigenvector orthogonality $\psi_0 \perp \psi_-$ establishes:

$$w^T v_1 = \sum_{i \in V} w_i = \sum_{i \in V_1} \psi_{0,i}^2 - \sum_{i \in V_2} \psi_{0,i}^2 = \psi_0^T \psi_- = 0$$

Therefore, the L_2 mass of the Perron vector is split equally between the two partition classes. Biorthogonality constraints for the dominant subspace are satisfied:

$$u_1^T v_2 = \sum_{i \in V_1} \psi_{0,i}^2 - \sum_{i \in V_2} \psi_{0,i}^2 = 0, \quad w^T v_2 = \sum_{i \in V} \psi_{-,i}^2 = \sum_{i \in V} \psi_{0,i}^2 = 1$$

B. Eigenvalue Decay and Asymptotic Oscillation

By the Perron-Frobenius theorem, all remaining subdominant eigenvalues satisfy $|\mu_\alpha| < 1$ for $\alpha \geq 3$. State evolution from an arbitrary initial distribution $\pi(0)$ follows:

$$\pi(t) = (\pi(0)v_1)(1)^t \pi^* + (\pi(0)v_2)(-1)^t w^T + \sum_{\alpha \geq 3} (\pi(0)v_\alpha) \mu_\alpha^t u_\alpha^T$$

Taking $t \rightarrow \infty$, decaying modes vanish exponentially ($|\mu_\alpha|^t \rightarrow 0$), yielding the leading asymptotic form:

$$\pi_i(t) = \psi_{0,i}^2 + c(-1)^t \sigma_i \psi_{0,i}^2 + o(1)$$

where $\sigma_i = +1$ for $i \in V_1$ and $\sigma_i = -1$ for $i \in V_2$. Hence, for the unmodified discrete-time chain, $\lim_{t \rightarrow \infty} \pi(t) = \pi^*$ if and only if $c = 0$.

For a point-mass distribution localized at node $k \in V_1$ ($\pi(0) = e_k^T$), we have $c_1 = 1$ and $c = +1$. The sequence alternates between two disjoint limits:

$$\lim_{\substack{t \rightarrow \infty \\ t \text{ even}}} \pi_i(t) = \begin{cases} 2\psi_{0,i}^2 & \text{if } i \in V_1 \\ 0 & \text{if } i \in V_2 \end{cases}, \quad \lim_{\substack{t \rightarrow \infty \\ t \text{ odd}}} \pi_i(t) = \begin{cases} 0 & \text{if } i \in V_1 \\ 2\psi_{0,i}^2 & \text{if } i \in V_2 \end{cases}$$

Initializing at node $k \in V_2$ yields $c = -1$, producing an identical, phase-inverted limit cycle. Although the chain is periodic, irreducibility and positive recurrence guarantee that the Cesaro time average $\frac{1}{T} \sum_{t=0}^{T-1} \pi(t) \rightarrow \pi^*$, and the pairwise parity average $\frac{1}{2}[\pi(t) + \pi(t+1)] \rightarrow \pi^*$ as $t \rightarrow \infty$.

IV. Conclusion

The universal statement that the discrete-time MERW distribution relaxes to $\pi^* = \psi_0^2$ is not valid on finite connected bipartite graphs. The stationary profile π^* remains correct and is recovered by Cesaro or pairwise parity averaging. However, the instantaneous distribution converges to π^* only when the initial parity imbalance c vanishes; for $c \neq 0$, the distribution approaches a permanent period-two limit cycle. To obtain ordinary convergence of the instantaneous distribution, one must modify the dynamics, for example by using an aperiodic lazy walk $P_\alpha = (1 - \alpha)I + \alpha P$ ($0 < \alpha < 1$) or transitioning to continuous-time dynamics with generator $Q = P - I$. A companion numerical simulation is provided in the Supplemental Material file `maximal_entropy_random_walk_simulation.py` [2].

References

- [1] Z. Burda, J. Duda, J.-M. Luck, and B. Waclaw, Localization of the maximal entropy random walk, *Phys. Rev. Lett.* 102, 160602 (2009).
- [2] R. Fisher, Maximal entropy random walk periodicity simulation, (2026).